GENERALIZED MIXED FRACTIONAL BROWNIAN MOTION AS A GENERALIZED WHITE NOISE FUNCTIONAL

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ABSTRACT

In this paper we study the generalized mixed fractional Brownian motion in the white noise analysis framework, in particular how to realize and analyze generalized mixed fractional Brownian motion in the white noise space. Explicit expressions for the S-transform of the generalized mixed fractional Brownian motion and for its distributional derivative are also obtained.

Keywords: *generalized mixed fractional Brownian motion, white noise analysis*

ABSTRAK

Dalam tulisan ini kita mempelajari gerak Brown fraksional campur tergeneralisir dalam kerangka analisis white noise, khususnya bagaimana mewujudkan dan menganalisis gerak gerak Brown fraksional campur tergeneralisir di ruang white noise. Ekspresi eksplisit untuk transformasi S dari gerak Brown berperingkat campur tergeneralisir dan derivatif distribusional juga diperoleh.

Kata kunci: *gerak Brown berperingkat campur tergeneralisir, analisis white noise*

INTRODUCTION

It is well known that the fractional Brownian motion of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \left(B_t^H\right)_{t\geq 0}$ defined on some probability space $\left(\Omega, F, P\right)$ with the covariance function $E(B_t^H B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$. This generalization of Brownian motion was introduced by Mandelbrot and Van Ness in (Mandelbrot & Van Ness, 1968) and satisfies the following properties:

- $B_0^H = 0$ almost surely,
- its variance $E\left(\left(B_t^H\right)^2\right) = t^{2H}$ for all $t \geq 0$,
- B^H is self-similar of order H and its increments are stationary,
- the sample paths of B^H are almost surely Hölder continuous of order $\gamma < H$, and nowhere differentiable.

The first application of this process was made by climatologist H.E. Hurst in 1951 who used it to model the long term storage capacity of reservoir along the Nile river. Nowadays fractional Brownian motions have been widely accepted in mathematical modeling in science and engineering such as hydrology, telecommunication traffic, queueing theory and mathematical finance, see for example (Chakravarti & Sebastian, 1997; Hu & Oksendal, 2003; Leland, Taqqu, Willinger, & Wilson et al, 1994; Scheffer, 2001).

Cheridito in (Cheridito, 2001) generalized the concept of fractional Brownian motion to the so-called mixed fractional Brownian motion. Let a and b be two real numbers such that $(a,b) \neq (0,0)$.

Definition 1.1.

A mixed fractional Brownian motion (MFBM) of parameter H, a, and b is a stochastic process $M^H = \left(M_t^H\right)_{t\geq 0} = \left(M_t^{H,a,b}\right)_{t\geq 0}$ defined on some probability space $\left(\Omega, F, P\right)$ by

$$M_t^H = M_t^{H,a,b} = aB_t + bB_t^H$$

where $\left(B_t\right)_{t\geq 0}$ is a Brownian motion and $\left(B_t^H\right)_{t\geq 0}$ is an independent fractional Brownian motion of Hurst parameter H.

This process was introduced to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. The model is the process $\left(X_t^{H,a,b}\right)_{t\in[0,1]}$ defined by $X_t^{H,a,b} = X_0^{H,a,b} \exp\left(\upsilon t + \sigma M_t^{H,a,b}\right)$, where υ, σ are real constants, $a > 0, b = 1$, and $M_t^{H,a,b}$ is a MFBM of parameter H, a, and b. Zili in (Zili, 2006) proved some stochastic and analytic properties of MFBM. Another application of MFBM to the computer network traffic was investigated in the recent paper (Filatova, 2008).

Recently MFBM has been further generalized by Thäle in (Thäle, 2009) to the generalized mixed fractional Brownian motion. Let $\alpha_1,\ldots,\alpha_n,\ n\in N$ be real numbers and not all equal to zero.

Definition 1.2.

A generalized mixed fractional Brownian motion (GMFBM) of parameter $H=(H_1,\ldots,H_n)$, *and* $\alpha=(\alpha_1,\ldots,\alpha_n)$ *is a stochastic process* $Z^H=\left(Z_t^H\right)_{t\ge 0}=\left(Z_t^{H,\alpha}\right)_{t\ge 0}$ *defined on some probability space* (Ω,F,P) *by*

$$Z_t^H = Z_t^{H,\alpha} = \sum_{k=1}^{n}\alpha_k B_t^{H_k}$$

where $\left(B_t^{H_k}\right)_{t\ge 0}$ *are independent fractional Brownian motions of Hurst parameter* H_k, $k=1,\ldots,n$.

We collect some properties of the GMFBM. For additional information and proofs see (Thäle, 2009).

Proposition 1.3. (Thäle, 2009)

The GMFBM $Z^H=\left(Z_t^{H,\alpha}\right)_{t\ge 0}$ *is a centered Gaussian process with variance* $\sum_{k=1}^{n}\alpha_k^2 t^{2H_k}$ *and covariance function* $E(Z_t^{H,\alpha}Z_s^{H,\alpha})=\frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\left(t^{2H_k}+s^{2H_k}-|t-s|^{2H_k}\right)$. Z^H *has stationary increments and these increments are correlated if and only if* $H_k=\frac{1}{2}$ *for all* k. Z^H *is also* $s_{(c_1,\ldots,c_n;H_1,\ldots,H_n)}$ *−self similar process in the sense that* $\sum_{k=1}^{n}\alpha_k c_k^{-H_k}B_{c_k t}^{H_k}=\sum_{k=1}^{n}\alpha_k B_t^{H_k}$ *in distribution.* Z^H *is neither a Markov process nor a martingale, unless* $H_k=\frac{1}{2}$ *for all* k. Z^H *exhibits a long range dependence if and only if there exists* k *with* $H_k>\frac{1}{2}$. *For all* $T>0$, *with probability one* Z^H *has a version, the sample paths of which are Hölder continuous of order* $\gamma<\min_{1\le k\le n}H_k$ *on the interval* $[0,T]$. *Every sample path of* Z^H *is almost surely nowhere differentiable.*

The paper is organized as follow. In section 2 we review the necessary background of white noise analysis and construct a representation of a GMFBM in the white noise space. In section 3 we show that GMFBM is differentiable in some distribution space and its derivative is a generalization of the classical white noise process.

Generalized Mixed Fractional Brownian Motion in the White Noise Space

Let (Ω,B,μ) be the white noise space, i.e. Ω is the space of tempered distribution $S'(R)$, B is the Borel σ-algebra on $S'(R)$, and the probability measure μ is uniquely determined by the Bochner-Minlos theorem such that

$$\int_{S'(R)}\exp\left(i\langle\omega,f\rangle\right)d\mu(\omega)=\exp\left(-\tfrac{1}{2}|f|_0^2\right) \tag{1}$$

for all smooth rapidly decreasing function $f \in S(R)$. Here $\langle \omega, f \rangle$ denotes the dual pairing between $\omega \in S'(R)$ and $f \in S(R)$, and $|\,.\,|_0$ is the usual norm in $L^2(R)$. The corresponding inner product in $L^2(R)$ is denoted by $(.,.)_0$.

From (1) we can deduce that $\langle ., f \rangle$ is a centered Gaussian random variable with variance $|f|_0^2$. Because of the isometry

$$E_\mu\left(\langle ., f \rangle^2\right) = |f|_0^2, \quad f \in S(R)$$

we can extend $\langle ., g \rangle$ to $g \in L^2(R)$. Hence we have for $f, g \in L^2(R)$

$$E_\mu\left(\langle ., f \rangle \langle ., g \rangle\right) = (f, g)_0 \tag{2}$$

From (2) it follows that a continuous version of $\langle ., 1_{[0,t)} \rangle$, which exists by the Kolmogorov-Centsov theorem, is a standard Brownian motion B_t in the white noise space. Because every $f \in L^2(R)$ can be approximated arbitrarily close by step functions we have

$$\langle ., f \rangle = \int_R f(t)\, dB_t \tag{3}$$

where $\int_R f(t)\, dB_t$ denotes the classical Wiener integral of a function $f \in L^2(R)$.

Now we summarize the construction of a fractional Brownian motion with arbitrary Hurst parameter $H \in (0,1)$ in the white noise space. This construction was introduced by Bender in (Bender, 2003). More explanations and detail of proofs can be found in (Bender, 2003) and references therein. As GMFBM is a linear combination of independent fractional Brownian motions, its realization in the white noise space can be easily derived. Mandelbrot and Van Ness in (Mandelbrot & Van Ness, 1968) proved that for $H \in (0,1) \setminus \{\frac{1}{2}\}$ a fractional Brownian motion is given by a continuous version of the following Wiener integral

$$B_t^H = \frac{K_H}{\Gamma(H + \frac{1}{2})} \int_R \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dB_s$$

where K_H is the normalizing constant. Here Γ denotes Gamma function and $(x)_+$ denotes the positive part of x.

We will use fractional integral and fractional derivative to obtain a representation of fractional Brownian motion in terms of the indicator function. First, for $H \in (\frac{1}{2}, 1)$ we use fractional integrals of Weyl's type. Let $\beta \in (0,1)$, define

$$\left(I_-^\beta f\right)(x) := \frac{1}{\Gamma(\beta)} \int_x^\infty f(t)(t-x)^{\beta-1}\, dt = \frac{1}{\Gamma(\beta)} \int_0^\infty f(x+t) t^{\beta-1}\, dt$$

and

$$\left(I_+^\beta f\right)(x) := \frac{1}{\Gamma(\beta)} \int_{-\infty}^x f(t)(x-t)^{\beta-1}\, dt = \frac{1}{\Gamma(\beta)} \int_0^\infty f(x-t) t^{\beta-1}\, dt$$

if the integrals exists for all $x \in R$. Now for case $H \in (0, \frac{1}{2})$ we make use of fractional derivatives of Marchaud's type which for $\beta \in (0,1)$ and $\varepsilon > 0$ is given by

$$\left(D_{\pm}^{\beta} f\right)(x) := \frac{\beta}{\Gamma(1-\beta)} \lim_{\varepsilon\to 0^{+}} \int_{\varepsilon}^{\infty} \frac{f(x)-f(x\mp t)}{t^{\beta+1}}\,dt$$

if the limit exists.

Hence by (3) and Kolmogorov-Centsov theorem we have the following.

Theorem 2.1. (Bender, 2003)

For $H \in (0,1)$ define operator $N_{\pm}^{H}$ as

$$N_{\pm}^{H} f := \begin{cases} K_H D_{\pm}^{-\left(H-\frac{1}{2}\right)} f & \text{if } H \in (0,\tfrac{1}{2}) \\ f & \text{if } H = \tfrac{1}{2} \\ K_H I_{\pm}^{H-\frac{1}{2}} f & \text{if } H \in (\tfrac{1}{2},1) \end{cases}$$

Then a fractional Brownian motion with Hurst parameter H in white noise space is given by a continuous version of $\left\langle .,N_{-}^{H} 1_{[0,t)}\right\rangle$.

Thus the following representation of GMFBM is well defined.

Definition 2.2.

For $H = \left(H_1,\ldots,H_n\right)$, and $\alpha = \left(\alpha_1,\ldots,\alpha_n\right)$, $H_k \in (0,1)$, $\alpha_k \in R$, $n \in N$ a GMFBM of parameter H and α in the white noise space is given by the continuous version of $\left\langle .,\sum_{k=1}^{n} \alpha_k N_{-}^{H_k} 1_{[0,t)}\right\rangle$.

The following proposition gives some simple properties of the operator $N_{\pm}^{H}$.

Proposition 2.3. (Bender, 2003)

Let $H \in (0,1)$ and $f \in S(R)$. Then

- $\left(f, N_{-}^{H} 1_{[0,t)}\right)_0 = \int_0^t \left(N_{+}^{H} f\right)(s)\,ds$
- $N_{+}^{H} f$ is continuous
- $\left(f, N_{-}^{H} 1_{[0,t)}\right)_0$ is differentiable and $\frac{d}{dt}\left(f, N_{-}^{H} 1_{[0,t)}\right)_0 = N_{+}^{H} f(t)$

Generalized Mixed Fractional White Noise

As we know a GMFBM Z^H is nowhere differentiable on almost every path. However, we are going to show that Z^H is differentiable as a mapping from R into a space of stochastic generalized functions, the so-called Hida distributions. The distributional derivative of GMFBM is called generalized mixed fractional white noise.

According to Wiener-Ito decomposition theorem every $\varphi \in (L^2) := L^2(S'(R), B, \mu)$ can be decomposed uniquely as

$$\varphi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f_n \rangle, \quad f_n \in \hat{L}_c^2(R^n) \tag{4}$$

where $\hat{L}_c^2(R^n)$ denotes the space of symmetric complex-valued L^2 -functions on R^n, and $:\cdot^{\otimes n}:$ denotes the n tensor power of Wick ordered monomial. The above decomposition is called the Wiener-Ito expansion of φ. Moreover the (L^2)-norm $\|\varphi\|_0$ of φ is given by

$$\|\varphi\|_0^2 := E_\mu(\varphi^2) = \sum_{n=0}^{\infty} n! |f_n|_0^2 \tag{5}$$

Now consider Hamiltonian of harmonic oscillator $A := -\frac{d^2}{dx^2} + x^2 + 1$ and we define its second quantization operator $\Gamma(A)$ in terms of the Wiener-Ito expansion. The domain of $\Gamma(A)$, denoted by $D(\Gamma(A))$, is the space of function φ of the form (4) such that $f_n \in D(A^{\otimes n})$ and $\sum_{n=0}^{\infty} n! |A^{\otimes n} f_n|_0^2 < \infty$. Then, we define

$$\Gamma(A)\varphi := \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, A^{\otimes n} f_n \rangle, \quad \varphi \in D(\Gamma(A)).$$

Both operators, A and $\Gamma(A)$, are densely defined on $L^2(R)$ and (L^2), respectively. Furthermore they are invertible and the inverse operators are bounded.

For $p \in N_0$ and $\varphi \in D(\Gamma(A)^p)$ we define a more general norm as follow

$$\|\varphi\|_p := \|\Gamma(A)^p \varphi\|_0.$$

Now define

$$(S)_p := \{\varphi \in (L^2) : \Gamma(A)^p \textit{ exists and } \Gamma(A)^p \varphi \in (L^2)\}$$

and endow $(S)_p$ with the norm $\|\cdot\|_p$. If we define

$$(S) := \text{ projective limit of } \{(S)_p : p \in N_0\}$$

then (S) is a nuclear space and it is called the space of Hida test function. The topological dual $(S)^*$ of (S) is called the space of Hida distribution. It can be shown that

$$(S)^* = \bigcup_{p \geq 0} (S)_p^*$$

and the norms on the dual space $(S)_p^*$ of $(S)_p$ is given by

$$\|\varphi\|_{-p} := \|\Gamma(A)^{-p} \varphi\|_0, \quad p \in N_0.$$

Hence we arrive at the Gelfand triple $(S) \subset (L^2) \subset (S)^*$.

Dual pairing of $\Phi \in (S)^*$ and $\varphi \in (S)$ is denoted by $\langle\langle \Phi, \varphi \rangle\rangle$. If $\Phi \in (L^2)$ then

$$\langle\langle \Phi, \varphi \rangle\rangle = E_\mu(\Phi.\varphi) \tag{6}$$

For a complete discussion about (S) and $(S)^*$ see (Hida et al,1993; Kuo,1996).

Definition 3.1.

Let $I \subset R$ be an interval. A mapping $X : I \to (S)^*$ is called a stochastic distribution process. A stochastic distribution process X is said to be differentiable if the limit $\lim_{h\to 0} \frac{X_{t+h} - X_t}{h}$ exists in $(S)^*$. Note that convergence in $(S)^*$ means convergence in the inductive limit topology.

Now we are in the position to show that GMFBM Z^H is a differentiable stochastic distribution process. For $n \in N_0$ let ξ_n be the nth Hermite function. First, recall that tempered distribution space $S'(R)$ can be reconstructed as an inductive limit as follow. Define a family of norms on $L^2(R)$ by

$$|f|_{-p}^2 := \left|A^{-p} f\right|_0^2 = \sum_{k=0}^{\infty} (2k+2)^{-2p} \left(f, \xi_k\right)_0^2, \quad p \in N .$$

The last equation follows from the fact that ξ_k is an eigenfunction of A with eigenvalue $2k+2$. Then $S'(R)$ is the inductive limit of $S_{-p}(R)$, $p \in N$ which is defined as the completion of $L^2(R)$ with respect to $|.|_{-p}$. Note that convergence in the inductive limit topology coincides with both the convergence in the strong and the weak-$*$ topology of $S'(R)$.

Lemma 3.2. (Bender, 2003)

Let $H \in (0,1)$. Then $N_-^H 1_{[0,\cdot)} : R \to S'(R)$ is differentiable and

$$\frac{d}{dt} N_-^H 1_{[0,t)} = \sum_{k=0}^{\infty} \left(N_+^H \xi_k\right)(t)\xi_k .$$

From the representation $B_t^H = \left\langle \cdot, N_-^H 1_{[0,t)} \right\rangle$ the preceding lemma might suggest that

$$\frac{d}{dt} B_t^H = \left\langle \cdot, \sum_{k=0}^{\infty} \left(N_+^H \xi_k\right)(t)\xi_k \right\rangle.$$

Now the integrand in this Wiener integral is no longer an element of $L^2(R)$ but a tempered distribution. From (3) and isometry (2) we obtain the following isometry

$$\left\|\langle \cdot, f \rangle\right\|_{-p} = \left\|\langle \cdot, A^{-p} f \rangle\right\|_0 = \left|A^{-p} f\right|_0 = |f|_{-p} \tag{7}$$

Thus the Wiener integral can be extended to $f \in S_{-p}(R)$ such that the isometry (7) holds, and consequently to $f \in S'(R)$. Note that this extended Wiener integral is a Hida distribution and need not to be a random variable. The following theorem enables us to calculate the derivative of B^H .

Theorem 3.3. (Bender, 2003)

Let $I \subset R$ be an interval and let $F : I \to S'(R)$ be differentiable. Then $\langle \cdot, F(t) \rangle$ is a differentiable stochastic distribution process and

$$\frac{d}{dt}\langle \cdot, F(t)\rangle = \left\langle \cdot, \frac{d}{dt}F(t)\right\rangle.$$

Combining this theorem with Lemma 3.2 we see that B^H is differentiable for $H \in (0,1)$ and

$$\frac{d}{dt}B_t^H = \left\langle \cdot, \sum_{k=0}^{\infty}\left(N_+^H \xi_k\right)(t)\xi_k \right\rangle.$$

Now for $t \in R$ we define the distribution

$$\left\langle \delta_t \circ N_+^H, f \right\rangle := \left(N_+^H f\right)(t),$$

where δ_t is Dirac delta function at t. Then

$$\left|\sum_{k=0}^{\infty}\left(N_+^H \xi_k\right)(t)\xi_k - \delta_t \circ N_+^H\right|_{-1}^2 = \sum_{k=0}^{\infty}\left(2n+2\right)^{-2}\left(\sum_{k=0}^{\infty}\left(N_+^H \xi_k\right)(t)\left(\xi_k,\xi_n\right)_0 - \left\langle \delta_t \circ N_+^H, \xi_n\right\rangle\right)^2$$
$$= 0$$

Hence we have

$$\frac{d}{dt}B_t^H = \left\langle \cdot, \delta_t \circ N_+^H \right\rangle.$$

Definition 3.4.

Let $H = \left(H_1,\ldots,H_n\right)$, *and* $\alpha = \left(\alpha_1,\ldots,\alpha_n\right)$, $H_k \in (0,1)$, $\alpha_k \in R$, $n \in N$. *Then the derivative of* $Z_t^{H,\alpha}$ *in* $\left(S\right)^*$

$$W_t^{H,\alpha} := \left\langle \cdot, \sum_{k=1}^{n}\alpha_k \delta_t \circ N_+^{H_k}\right\rangle$$

is called generalized mixed fractional white noise (GMFWN).

Note that this is really a generalization of the classical white noise $\left\langle \cdot, \delta_t \right\rangle$.

One of the fundamental tools in white noise analysis is the S-transform.

Definition 3.5.

For $\Phi \in \left(S\right)^*$ *the S-transform is defined by*

$$\left(S\Phi\right)(\eta) := \left\langle\left\langle \Phi, :\exp\left(\left\langle \cdot, \eta\right\rangle\right): \right\rangle\right\rangle, \quad \eta \in S\left(R\right).$$

The S-transform is well defined, because the Wick exponential

$$:\exp\left(\langle \cdot, \eta\rangle\right): = \exp\left(\langle \cdot, \eta\rangle - \tfrac{1}{2}|\eta|_0^2\right)$$

of the Wiener integral of a smooth rapidly decreasing function is a Hida test function. The S-transform also gives a convenient way to characterize an element in $\left(S\right)^*$, see (Kuo, 1996). Note that by definition if $X : I \to (S)^*$ is a differentiable stochastic distribution process, then

$S\left(\frac{d}{dt}X_t\right)(\eta)=\frac{d}{dt}\left(SX_t(\eta)\right)$. Now we can obtain the explicit expressions of the S-transform of the GMFBM and GMFWN.

Proposition 3.6.

Let $H\in(0,1)$. *Then for every* $\eta\in S(R)$

- $\left(SZ_t^{H,\alpha}\right)(\eta):=\sum_{k=1}^{n}\alpha_k\left(\eta,N_-^{H_k}1_{[0,t)}\right)_0$
- $\left(SW_t^{H,\alpha}\right)(\eta):=\sum_{k=1}^{n}\alpha_k\left(N_+^{H_k}\eta\right)(t)$

Proof.

- By (6) and the polarization of (5) we get

$$\left(SZ_t^{H,\alpha}\right)(\eta):=\left\langle\left\langle Z_t^{H,\alpha},:\exp\left(\langle\cdot,\eta\rangle\right):\right\rangle\right\rangle=E_\mu\left(\left\langle\cdot,\sum_{k=1}^{n}\alpha_kN_-^{H_k}1_{[0,t)}\right\rangle:\exp\left(\langle\cdot,\eta\rangle\right):\right)$$

$$=\sum_{k=1}^{n}\alpha_k\left(\eta,N_-^{H_k}1_{[0,t)}\right)_0.$$

- This immediately follows from Proposition 2.3 and the fact that $W_t^{H,\alpha}$ is the derivative of $Z_t^{H,\alpha}$.

CONCLUSION

A representation of generalized mixed fractional Brownian motion in white noise space has been constructed. We also show that this stochastic distribution process is differentiable in distributional sense. Moreover we provide the formula for the S-transform of these processes. In the white noise analysis framework we are interested in the Donsker's delta of GMFBM, self-intersection local times of GMFBM, generalization to arbitrary spatial dimension, stochastic integral with respect to GMFBM and stochastic differential equations driven by GMFBM as further investigations.

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